INTRODUCTION

In recent years, several finite element models have been developed for reinforced concrete slabs. The models have been of both the modified stiffness (2,12) and the layered (10,11,14,17) types. The modified stiffness models have had only limited success and have not proved to be applicable to a wide range of structures. Layered models, made up of a series of layers idealized as being in a state of plane stress, have met with greater success. In the layered models, the material idealizations of concrete have been either elastic or elasto-plastic in compression, and cracking has been limited to distinct levels within the slab. However, gradual softening in compression and continuous cracking have not been represented, resulting in a loss of realism in load-deflection behavior.

This paper describes a nonlayered finite element model for reinforced concrete slabs that includes the nonlinear variation of material properties through the depth of the slab. Reinforced concrete slabs are modeled as incrementally elastic, anisotropic plates. Axes of anisotropy coincide with the slab yield lines. Concrete is modeled as a nonlinear material in compression and as a linear brittle material in tension. Steel is represented as a uniaxial material with a bilinear stress-strain curve. Bond slip between steel and concrete, creep, shrinkage, temperature, long-term loading, cyclic loading, membrane stresses, and strength variation due to biaxial stresses are not included. Loads are applied incrementally, and the solution is corrected using successive iterations.

The paper briefly examines key aspects of the slab model. The anisotropic plate equations are presented in general form, and as they apply to reinforced concrete slabs. The material representations and moment curvature relationships are described. The numerical procedure is outlined and the model is compared with experimental results for two beams and three slabs. Additional examples are given in Ref. 1.

Note.—Discussion open until June 1, 1978. To extend the closing date one month, a written request must be filed with the Editor of Technical Publications, ASCE. This paper is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 104, No. ST1, January, 1978. Manuscript was submitted for review for possible publication on April 4, 1977.

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Material Axes.—The anisotropic properties of reinforced concrete slabs are due primarily to cracking concrete and yielding steel. For orthotropically reinforced concrete slabs, the crack directions (yield lines) do not necessarily coincide with the principal moments or with the reinforcing steel. Only in isotropically reinforced slabs do the directions of the cracks and principal moments coincide.

In this study, yield line theory is used to establish the material axes during the solution procedure, both before and after yield lines form.

Following the work of Kemp (13), the orientation of the yield lines (Fig. 1) is given in terms of the principal bending moments per unit width, $M'_y$ and $M'_x$:

$$
\psi = \frac{1}{2} \tan^{-1} \left( \frac{(M'_x - M'_y) \sin 2\phi}{(M'_y - M'_x) - (M'_x - M'_y) \cos 2\phi} \right)
$$

in which $\psi$ is the angle between the normal to the principal moment, $X'$, and the normal to the yield line direction, $l$; and $\phi$ is the angle between the normal to the principal moment, $X'$, and the steel direction, $X$.

Yield lines form when the following criterion is met:

$$
M'_x M'_y (\sin^2 \phi + \mu \cos^2 \phi) + M'_x M'_y (\cos^2 \phi + \mu \sin^2 \phi) - M'_x M'_y - \mu M'_y^2 = 0
$$

in which $\mu = M'_y / M'_x$.

During the solution process, rotation of the material axes within each element is permitted before yielding occurs in the reinforcing steel. The crack directions are determined using Eq. 1. After yielding, the material axes are fixed. Twisting moments will, in general, exist on the yield lines. Tests show that the reorientation, or kinking, of reinforcing bars at the yield lines can be neglected (5).
Concrete.—Concrete is modeled as an incrementally linear orthotropic material and is represented using the differential stress-strain relations developed by Darwin and Pecknold (6,7,8):

$$\begin{align*}
\frac{d\sigma_1}{d\sigma_2} &= \frac{1}{1 - \nu^2} \begin{bmatrix}
E_1 & \nu \sqrt{E_1 E_2} & 0 \\
\nu \sqrt{E_1 E_2} & E_2 & 0 \\
0 & 0 & \frac{1}{4} (E_1 + E_2 - 2\nu \sqrt{E_1 E_2})
\end{bmatrix} \begin{bmatrix}
\frac{d\varepsilon_1}{d\varepsilon_2} \\
\frac{d\varepsilon_2}{d\gamma_{12}}
\end{bmatrix}
\end{align*}$$

(3)

The tangent moduli, $E_1$ and $E_2$, and $\nu$, the "equivalent" Poisson's ratio ($= \sqrt{\nu^2}$), are stress dependent.

In plate bending problems, the state of stress is essentially two-dimensional, and the strain in one direction is a function, not only of the stress in that direction, but also of the stress in the orthogonal direction, due to the Poisson effect. For the proposed nonlinear model it is convenient to analyze the two material directions independently, keeping track of the portion of the strain in each direction that controls the nonlinear behavior of the concrete.

The device of "equivalent uniaxial strain," developed by Darwin and Pecknold (6,7,8) is used to represent this portion of the strain on the material axes. Equivalent uniaxial strain is expressed incrementally as follows:

$$\varepsilon_{iu} = \sum_{\text{load increments}} \frac{\Delta\varepsilon_{iu}}{E_i}, \quad i = 1, 2, \ldots$$

(4)

in which $\Delta\sigma_i$ is the incremental change in stress, $\sigma_i$; and $E_i$ represents the tangent modulus in the $i$ direction at the start of the load increment. The concept is explained more fully in Refs. 6, 7, and 8.

Equivalent uniaxial strains are not transformable, as are true strains. However, for a fixed set of axes, equivalent uniaxial strains may be converted to true...
strains, \( \epsilon_1 \) and \( \epsilon_2 \), which are transformable. Thus

\[
\begin{align*}
\epsilon_1 &= \sum \text{load increments} \left\{ \Delta \epsilon_{1n} - \nu \left( \frac{E_2}{E_1} \right)^{1/2} \Delta \epsilon_{2n} \right\} \\
\epsilon_2 &= \sum \text{load increments} \left\{ -\nu \left( \frac{E_1}{E_2} \right)^{1/2} \Delta \epsilon_{1n} + \Delta \epsilon_{2n} \right\}
\end{align*}
\]

The nonlinear behavior of concrete in compression is represented using the equation suggested by Saenz (16), shown in Fig. 2. Concrete is treated as a linear brittle material in tension. The details of the stress-strain curve are presented in Appendix I.

Steel.—Steel is idealized as an elasto-plastic, uniaxial material, as shown in Fig. 3.

**Flexural Stiffness.**—The orthotropic concrete and the uniaxial steel representations are combined in the plate bending equations to form an anisotropic plate in which the material properties vary through the depth of the plate. The axes of anisotropy and the orthotropic axis of the concrete coincide with the slab yield lines. Incremental changes in moment are related to incremental changes in curvature on the axes of anisotropy as follows (1):

\[
\begin{align*}
\left\{ \Delta M_1 \right\} &= \begin{bmatrix} C_{11} Z_1 dZ & C_{12} Z_1 Z_2 dZ & C_{13} Z_1 Z_3 dZ \end{bmatrix} \Delta \kappa_1 \\
\left\{ \Delta M_2 \right\} &= \begin{bmatrix} C_{21} Z_2 dZ & C_{22} Z_2 Z_2 dZ & C_{23} Z_2 Z_3 dZ \end{bmatrix} \Delta \kappa_2 \\
\left\{ \Delta M_{12} \right\} &= \begin{bmatrix} \text{Sym} & C_{12} Z_1 Z_2 dZ & \text{Sym} \end{bmatrix} 2 \Delta \kappa_{12} \\
\text{or} \quad \left\{ \Delta M_1 \right\} &= \begin{bmatrix} D_{11} & D_{12} & D_{13} \\
D_{21} & D_{22} & D_{23} \\
\text{Sym} & \text{Sym} & \text{Sym} \end{bmatrix} \begin{bmatrix} \Delta \kappa_1 \\
\Delta \kappa_2 \\
2 \Delta \kappa_{12} \end{bmatrix}
\end{align*}
\]

\( C_{ij} \) represent the coefficients of the anisotropic constitutive matrix and are functions of depth as well as location in the plate; and \( Z_1, Z_2, (Z_{12} = (Z_1 + Z_2)/2) \) are measured from the instantaneous neutral axes in the 1 and 2 directions, respectively. For a given direction, the instantaneous neutral axis
is the point of zero incremental change in stress and is located at a distance \( \Delta Z \) from the neutral axis (Fig. 4):

\[
\Delta Z = \frac{\int E \varepsilon' \, dZ}{\int E \varepsilon \, dZ} \quad \text{(8)}
\]

The \( D \) matrix in Eq. 7 may be expressed as the sum of separate contributions for steel and concrete:

\[
D = D_s + D_c \quad \text{(9)}
\]

The steel contribution for a layer of orthogonal steel oriented at an angle, \( \theta \) (Fig. 1), with the \( 1 \) axis of the slab is

\[
D_{11} = (A_x E_x \cos^2 \theta + A_y E_y \sin^2 \theta) \frac{Z_1}{Z_2};
D_{12} = (A_x E_x + A_y E_y) \sin \theta \cos \theta \frac{Z_1}{Z_2};
D_{13} = (-A_x E_x \cos^2 \theta + A_y E_y \sin^2 \theta) \sin \theta \cos \theta \frac{Z_1}{Z_2};
D_{22} = (A_x E_x \sin^2 \theta + A_y E_y \cos^2 \theta) \frac{Z_2}{Z_1};
D_{23} = (-A_x E_x \sin^2 \theta + A_y E_y \cos^2 \theta) \sin \theta \cos \theta \frac{Z_2}{Z_1};
D_{33} = (A_x E_x + A_y E_y) \sin \theta \cos \theta \frac{Z_2}{Z_1} \quad \text{(10)}
\]

in which \( A_x, A_y \) = the areas per unit width of steel in the \( X \) and \( Y \) (steel) coordinates; \( E_x, E_y \) = the moduli of elasticity of the steel; \( Z_1, Z_2 \) = depths of steel layers measured from instantaneous neutral axes; and \( Z_{12} = (Z_1 + Z_2)/2 \).

The contribution of the concrete to flexural stiffness is

\[
D_{11} = \frac{1}{1 - \nu^2} \int E_z Z_1^2 \, dZ;
D_{12} = \frac{1}{1 - \nu^2} \int \sqrt{E_z E_z} Z_1 Z_2 \, dZ;
D_{22} = \frac{1}{1 - \nu^2} \int E_z Z_2^2 \, dZ;
D_{33} = \frac{1}{4(1 - \nu^2)} \int (E_z + E_\theta - 2\nu \sqrt{E_z E_\theta}) Z_1^3 \, dZ; \quad D_{13} = D_{23} = 0 \quad \text{(11)}
\]

in which \( Z_1, Z_2, \) and \( Z_{12} \) are defined following Eq. 7.

The \( D \) matrix in Eq. 7 is transformed to the global coordinates \((X, Y)\) and used to form the element stiffness matrix.

**Moment-Curvature Relationships.**—Moment-curvature relationships are used in the finite element representation to correct the linear load-deflection increments to match the nonlinear behavior of the slab. Using the relationships presented in the following, the slab moments are calculated during each step in the loading sequence as a function of the curvature and the material properties.

The moment-curvature equations are expressed in terms of “equivalent uniaxial curvature” (1), which is analogous to equivalent uniaxial strain. This concept allows the moment-curvature relationships on the yield lines to be treated independently (i.e., as if the slab were subjected to uniaxial bending on each material axis). Expressing the moment-curvature relations in terms of equivalent uniaxial curvature uncouples the two directions and allows a substantial simpli-
fication of a complex nonlinear interaction problem.

The equivalent uniaxial curvature is expressed in incremental form as

\[ \kappa_n = \sum_{\text{load increments}} \frac{M_i}{D_n} ; \quad i = 1, 2 \]  \hspace{1cm} (12)

in which \( M_i \) is the incremental change in the moment per unit width in the \( i \) direction during a load increment (from the finite element analysis); and \( D_n \) is the flexural stiffness in the \( i \) direction at the start of the load increment. An increment of equivalent uniaxial curvature, \( \Delta \kappa_n \), represents the change in curvature along the \( i \)-axis that would occur for a change in moment of \( \Delta M_i \), with \( \Delta M_i = 0 \).

In reinforced concrete slabs, the material axes do not, in general, coincide with the reinforcing steel. Therefore, the steel forces must be transformed to the material axes for use in the moment-curvature equation:

\[ f_i = f_1 \cos^2 \theta + f_2 \sin^2 \theta ; \quad f_2 = f_1 \sin^2 \theta + f_2 \cos^2 \theta \]  \hspace{1cm} (13)

in which \( f_1, f_2 \) are the components of the steel forces in the material (1, 2) axes; \( f_1, f_2 \) are the steel forces per unit width in the steel \( (X, Y) \) axes; and \( \theta \) is the angle between the steel and the material axes.

The equilibrium equations on the material axes are presented in terms of concrete stresses, equivalent uniaxial strains and curvatures and steel forces in Eq. 14. The equations are similar to those used by Vebo and Ghali (18). Thus

\[ \sum_i F = \frac{1}{\kappa_n} \int_{\text{limits of integration}} \sigma_i(e) d\varepsilon + \int_{\text{limits of integration}} f_i' - f_i = 0; \quad i = 1, 2 \]  \hspace{1cm} (14a)

\[ \sum_i M = \frac{1}{\kappa_n^2} \int_{\text{limits of integration}} \sigma_i(e) d\varepsilon + \int_{\text{limits of integration}} f_i' (kH - d_i') + f_i [(1 - k)H - d_i] - M_n = 0; \quad i = 1, 2 \]  \hspace{1cm} (14b)

in which \( \varepsilon = \) equivalent uniaxial strain at distance \( Z' \) from the neutral axis (see Fig. 4); \( \varepsilon_n, \varepsilon_m = \) equivalent uniaxial strains at the extreme fibers of the section, limits of integration; \( \sigma_i(e) = \) stress in the concrete at distance \( Z' \) from the neutral axis in the \( i \) direction; \( f_i, f_i' = \) components of the tension and compression steel forces per unit width in the \( i \) direction; \( d_i, d_i' = \) distances from extreme fibers to the centroids of the tension and compression steel, respectively; \( H = \) depth of the slab; \( kH = \) depth of the neutral axis; and \( M_n = \) the resisting moment in the \( i \) direction.

Eq. 14a is used to establish the location of the neutral axis for a given equivalent uniaxial curvature, \( \kappa_n \). The unknown in the equation is the concrete strain at the extreme compressive fiber, \( \varepsilon_c = \varepsilon_n \) or \( \varepsilon_m \), which together with the curvature, establishes the location of the neutral axis. The integrals in Eq. 14 are evaluated numerically, using Gaussian quadrature. The solution of Eq. 14a is considered to have converged if the change in the strain, \( \varepsilon_n \), is less than \( 10^{-7} \).

In order to solve Eq. 14a for \( \varepsilon_n \), the components of the steel forces, \( f_i \) and \( f_i' \), which are functions of \( f_1 \) and \( f_2 \) (and \( f_1', f_2' \) and \( f_1', f_2' \) (Eq. 13), must be known. The values of \( f_1 \), \( f_2 \), and \( \varepsilon_n \) are obtained using the technique summarized in the following.
### TABLE 1.—Material Properties of Test Specimens

<table>
<thead>
<tr>
<th>Investigators</th>
<th>Designation</th>
<th>Material Properties</th>
<th>X direction</th>
<th>Y direction</th>
<th>Reinf. Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaston, Sless, and Newmark</td>
<td>T1MA</td>
<td>4.6 3.86 46 28.2</td>
<td>0.068</td>
<td></td>
<td>p = 0.0062</td>
</tr>
<tr>
<td>Cardenas</td>
<td>T3MA</td>
<td>4.8 3.94 41 28.2</td>
<td>0.339</td>
<td></td>
<td>p = 0.0322</td>
</tr>
<tr>
<td>Sozen</td>
<td>B7</td>
<td>4.92 4.00 0.15</td>
<td>0.0352 0.0352</td>
<td>45°</td>
<td>p = 0.01</td>
</tr>
<tr>
<td>McNiece</td>
<td>Two-way slab</td>
<td>5.5 4.15 0.15</td>
<td>29.0 0.0111 0.0111</td>
<td>0°</td>
<td>p = 0.0085</td>
</tr>
</tbody>
</table>

*In kips per square inch.

*b In kips per square inch × 10^6.

*c $E_0 = 57000 \sqrt{f'_c}, \sigma_y = 7.5 \sqrt{f'_c}$ are used in the analysis for all problems.

*d In square inches per inch.

*Measured counterclockwise from the global coordinate system to the X-axis of the steel.

fAssumed, original data not available.

Note: 1 in. = 25.4 mm, 1 kip = 4.46 kN.

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**FIG. 5.—Load-Deflection Curves for Beam Test Specimens, T1MA and T3MA (9)** (1 in. = 25.4 mm; 1 kip = 4.45 kN)
For an iteration, the most recent values of the steel forces, $f_i$ and $f_{y}$, are used in Eq. 13 to approximate the components of the steel forces, $f_i (i = 1, 2)$. The equivalent uniaxial curvature, $\kappa_{eq}$, is known from the finite element solution. Eq. 14a is solved for $\epsilon_{eq}$ in the 1 and 2 directions using the Newton-Raphson technique. The equivalent uniaxial strains are converted to true strains, $\epsilon$, using Eq. 5. The true strains at the surface of the slab are transformed to the steel $(X, Y)$ coordinates. The values of $f_i$ and $f_y$ are obtained using these strains and the true curvatures. The updated steel forces are transformed back to the $(1, 2)$ axes to obtain the components, $f_{i}$, which are again used to solve Eq. 14a for $\epsilon_{eq}$. Convergence is obtained when the ratio of the change in steel force in each direction to the original value is less than 1%.

After convergence, the resisting moments, $M$, are obtained from Eq. 14b. In most cases, convergence is obtained in less than five iterations, but more iterations are required when excessive strains occur in the concrete or the steel.

FIG. 6.—Slab Test Specimens, B10 and B7 (5) (1 in. = 25.4 mm; 1 psi = 6.89 kN/m$^2$)
NUMERICAL PROCEDURE

Finite Element.—The finite element used in this study is a four-noded, rectangular plate bending element with 16 degrees-of-freedom, developed by Bogner, Fox, and Schmit (3). The element tangent stiffness and moment-curvature relations are calculated using the material properties at the center of each element.

Numerical Solution.—To obtain the load-deflection behavior of slabs to failure, loads are applied incrementally. Load increments are reduced in magnitude after cracking of the concrete or yielding of the steel, or both. Loads are corrected using the Initial Stress Method (19). The constant stiffness approach is used before cracks form in the slab using the initial stiffness of the structure. After cracks form, the stiffness matrix is updated following each iteration. Since only a portion of the structure may be softened by cracking, realistic load-deflec-

![Graph](image)

FIG. 7.—Moment-Curvature Curves for Slab Test Specimens, B10 and B7 (5) (1 in. = 25.4 mm; 1 kip = 4.45 kN)

tion behavior may be modeled only if the structure stiffness is recalculated to account for the cracks.

NUMERICAL EXAMPLES

General.—Five numerical examples are presented: Two singly reinforced concrete beams, tested by Gaston, Siess, and Newmark (9); two slabs subjected to uniform moment, tested by Cardenas and Sozen (5); and a slab supported on four corners, tested by McNeice (15). The results for the slabs are also compared with analytical results obtained by others (10,11,12,14,17).

The material properties of the test specimens are presented in Table 1.

Beams.—Gaston, Siess, and Newmark (9) conducted a series of tests on simply supported reinforced concrete beams. The beams were loaded at the third points. Load-deflection curves were obtained experimentally. Two of the beams, T1MA and T3MA (Fig. 5), are studied. T1MA is under-reinforced, while T3MA has a balanced reinforcement ratio.

The load deflection curves are shown in Fig. 5. The analytical solutions for both beams compare favorably with the experimental results and indicate that only small changes in the resisting moment occur after the steel begins to yield.
The stresses in the concrete reach a maximum immediately after yielding of the steel in the balanced beam and after a few additional load increments in the underreinforced beam. The concrete stresses at the extreme fibers then begin to decrease on the downward portion of the stress-strain curve.

Slabs under Uniform Moment.—In an experimental investigation of flexural yield criteria by Cardenas and Sozen (5), a series of slabs were subjected to uniaxial bending moments. The slabs were simply supported on two edges and free on the other two. The proposed model is used to simulate the moment-curvature behavior of two slabs, B7 and B10. The model also matches the change in steel strain and concrete strain with increasing moment (1). The slabs were isotropically reinforced, with slab B10 reinforced parallel to slab edges, and slab B7 reinforced at an angle of $45^\circ$ with the slab edges. One half of each slab is analyzed using three elements, as shown in Fig. 6.

The proposed model is compared with the results obtained experimentally by Cardenas and Sozen (5) and analytically by Hand, Pecknold, and Schnobrich (10,11) in Fig. 7.

Hand, Pecknold, and Schnobrich analyze B7 and B10 using a layered, 20-degree-of-freedom, shallow shell finite element. Each layer is assumed to be in a state of plane stress and the material properties are assumed to be constant over the layer thickness. Slabs B7 and B10 are represented by a single finite element.

The proposed model shows good agreement with the experimental results for moment-curvature and moment-strain behavior (1). Prior to cracking, the orientation of the steel has little effect on slab behavior. After cracking, the direction of the reinforcing steel has a significant effect on the load-deflection and moment-curvature curves. The greater the inclination of the steel direction with respect to the applied moment, the greater the deflections and curvatures. This is due to the reduced contribution of inclined steel to the flexural stiffness of a reinforced concrete slab (Eq. 10). As predicted by yield line theory, the orientation of steel in an isotropically reinforced slab has no influence on the ultimate loads. The same ultimate moments are obtained for both slabs, as may be seen in Fig. 7.

FIG. 8.—Two-Way Slab Supported at Corners (12,15) (1 in. = 25.4 mm; 1 psi = 6.89 kN/m$^2$)
Two-Way Slab.—The two-way slab tested by McNeice (15) is square, supported at the four corners, and reinforced with an isotropic mesh (Fig. 8). The slab is subjected to a central concentrated load.

This problem is of special interest: (1) It is a two-way slab with moments varying through the slab in two directions; and (2) this slab has been analyzed by several other investigators (11, 12, 14, 17), and comparison with their models can be made. Only one-quarter of the slab is considered, because of symmetry.

To analyze this slab, Jofriet and McNeice (12) use the modified stiffness approach and treat the steel and concrete as elastic materials. Nonlinear behavior is modeled by changing the slab stiffness during the application of load. The modulus of elasticity of a cracked section is reduced to 0.57E₀. Thirty-six quadrilateral plate bending elements are used.

For his analytical work, Scanlon (17) uses a layered rectangular plate bending element with four degrees-of-freedom at each corner node. Cracks are assumed to progress through the thickness of the element, layer by layer, parallel and perpendicular to the orthogonal reinforcement. Steel and concrete are taken as linear materials with no post-yield behavior or failure considered.

Lin and Scordelis (14) extend Scanlon's approach to include elasto-plastic behavior for steel and concrete. They account for the coupling effect between membrane and bending action. A triangular element with 15 degrees-of-freedom is used. Eighteen elements are used to represent one quarter of the slab.

Hand, Pecknold, and Schnobrich (11) model the slab using 36 elements.

For the proposed model, 16 elements are used. The deflection at point A, located 3 in. (76 mm) from the concentrated load (Fig. 8), is used to compare the analytical and experimental results. The deflection is obtained approximately from deflections at nodal points 20 and 25.

The proposed solution is compared with the experimental results and the other analytical models (Fig. 9). It is in good agreement with the experimental curve and with the solution of Jofriet and McNeice. The model provides a better match than the three layered models. The ability to represent cracking as a continuous process, not limited to distinct layers, is viewed as a strong
point of the proposed model. As demonstrated in Ref. 1, it also has the ability
to represent a wider range of test results than the model offered by Jofriet
and McNeice.

CONCLUSIONS

The proposed model and method of analysis give satisfactory results for
predicting the flexural behavior of reinforced concrete beams and slabs. The
ability to represent cracking as a continuous process appears to be a strong
point of the model. The softening of concrete in compression appears to be
important, but less critical. The numerical examples presented here and in Ref.
1 indicate that the effect of biaxial stresses on concrete stiffness and strength
is insignificant in modeling the behavior of reinforced concrete slabs under
monotonic load. Good matches with test data are obtained for the numerical
examples presented without modeling bond slip between steel and concrete or
kinking of the steel at the yield lines. Work during the study indicates that
reducing the size of the load increment helps to insure an accurate analysis
after cracking or yielding begin, or both. As used in this model, the yield line
theory proves to be an excellent tool, not only for predicting ultimate strength,
but for formulating the full load-deflection behavior of the slabs. The model
demonstrates analytically, that the orientation of steel in isotropically reinforced
slabs effects slab stiffness, but not strength. This conclusion matches experimental
observations.

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APPENDIX I.—UNIAXIAL STRESS-STRAIN CURVES

The equation suggested by Saenz (16) is adapted to model the nonlinear behavior
of concrete in compression:

\[
\sigma = \frac{\epsilon}{A + B \epsilon + C \epsilon^2 + D \epsilon^3}
\]

in which \( \epsilon \) is the strain at any point in the concrete section; and \( \sigma \) is the
stress in concrete corresponding to strain \( \epsilon \). The strain, \( \epsilon \), is the “equivalent
uniaxial strain,” \( \epsilon_u \). Parameters \( A \), \( B \), \( C \), and \( D \) are defined as follows:

\[
A = \frac{1}{E_u}; \quad B = \frac{R_e + R - 2}{R_e \sigma_u \epsilon_u}; \quad C = \frac{1 - 2R}{R_e \sigma_u \epsilon_u}; \quad D = \frac{R}{R_e \sigma_u \epsilon_u^2}
\]

\[
R = \frac{R_e(R_f - 1)}{(R_f - 1)^3} \quad \frac{1}{R_e}; \quad R_e = \frac{E_u \epsilon_u}{\sigma_u}; \quad R_f = \frac{\sigma_f}{\sigma_u}; \quad \text{and} \quad R_e = \frac{\sigma_f}{\epsilon_u}
\]
in which \( E_0 \) is the initial tangent modulus of elasticity as determined from uniaxial compression tests; the approximate formula given by ACI (318-71) (4) is used; \( \sigma_p \) is the maximum concrete strength (= \( f'_c \) for the uniaxial case); and \( \epsilon_p \) is the strain at which the peak compressive stress is attained. In this study its value varies between 0.0020 and 0.0021 for concrete strengths varying between 3,500 psi (24 MN/m²) and 5,700 psi (39 MN/m²) (1).

The maximum strain \( \epsilon_p = 4 \epsilon_c \), and \( \sigma_p \) is the strength at \( \epsilon_p \), approximated by (1):

\[
\sigma_p = 450 + 0.25 \sigma_c - 3.4 \times 10^{-3} \epsilon_c^2
\]

The tensile strength of concrete is approximated by the modulus of rupture, \( \sigma_t = 7.5 \sqrt{f'_c} \).

**APPENDIX II.—REFERENCES**


4. Building Code Requirements for Reinforced Concrete, 318-71, American Concrete Institute, 1971.


